Network Representation of Transcendental Impedance Functions

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The purpose of the paper is to show that the admittance or impedance of certain continuous structures, such as, for example, a finite length of transmission line of any sort, or resonant cavity, can be represented exactly at all frequencies by a network comprising lumps of constant resistance R, inductance L, conductance G and capacitance C. The network will contain an infinite number of branches, in general, although a finite number may be used if it is desired to represent only certain modes.

The procedure is based upon a proposition known to students of function theory as "Mittag-Leffler's theorem," which amounts, roughly, to an extension of rational functions to apply to transcendental functions of the type encountered in the theory of continuous structures.

Several illustrative examples of the network synthesis are given.

GENERAL

Students of network theory are familiar with the fact that the impedance at a pair of terminals in a linear network comprising a finite number of resistors, inductors and capacitors, connected in any manner, is a rational function of the frequency having, in general, the fractional form of one polynomial divided by another. They are also familiar with the partial fraction rule whereby the function can be broken up into a series of elementary fractions, each of which exhibits one of the poles of the original function. This form is sometimes useful in the problem of network synthesis, where the impedance function is given and the object is to find a network having this impedance.

The purpose of the present paper is to show how a similar procedure can be carried out for certain transcendental impedance functions pertaining to structures having distributed constants, such as, for example, a resonant cavity or a piece of transmission line. The method employs a well-known proposition of function theory, which is usually referred to as Mittag-Leffler's theorem. This theorem provides a tool for breaking up a transcendental meromorphic function into an infinite series of simple fractions in much the same way as the partial fraction rule is used to break up a rational meromorphic function. The series representation

provides a means of determining a network of resistors, inductors and capacitors that will have an impedance equal to the specified transcendental impedance function. This process will be referred to as obtaining a "network representation" of the function. If the given function is the impedance of some continuous (i.e., non-lumped) electric structure, the result will be an equivalent network for the structure. For other purposes, such as, possibly, analogue methods of computing, the given function may not arise from any electrical structure. In either case, the network representations to be derived are possible only if the function satisfies certain restrictions, which are stated in the section immediately following.

The discussion is confined to transcendental impedance functions because of the technological interest in the electromagnetic structures with which they are associated and because they have not received as much attention as rational functions in the literature dealing with network synthesis. The problem with which this paper is concerned can then be stated as follows: given, a transcendental impedance function satisfying certain conditions: to determine a network comprising elements of constant resistance, inductance and capacitance whose driving-point impedance function, at a pair of terminals, will equal the given function at all frequencies, real and complex (except at the poles).

For illustration of the procedure, three examples are given. The first is the impedance of a short-circuited or open-circuited transmission line in which the distributed primary constants, R, L, G and C are assumed to be invariable with frequency. The second and third examples are the impedances of resonant cavities driven in two different modes. In these examples the variation of resistance with frequency, due to "skin-effect," is taken into account.

IMPEDANCE FUNCTIONS

The functions under discussion will be referred to as "impedance functions" with the understanding that the term is meant to include "admittance functions" as well. By reason of the duality principle that runs through all electric circuit theory, any general proposition developed for one must apply to the other. The functional designation, $F(p)$, will be used to denote either an impedance or an admittance function. When a distinction is necessary, the impedance will be designated by $Z(p)$ and the admittance by $Y(p)$. The independent complex variable p is the generalized radian frequency. (For sustained sinusoidal currents and voltages, $p = i\omega = 2\pi if$ where f is the real frequency.)

For the applications contemplated, $F(p)$ is a transcendental mero-

morphic function, which term implies that the function is given by the ratio of two entire functions, one or both of which is transcendental, and that the singularities of the function are ordinary poles, except for the point at infinity, which is an essentially singular point. In order to realize the particular network developments to be given, it will be supposed that the function satisfies the further restrictions given below:

(1) All the poles lie in the left half of the p-plane with none on the imaginary axis.

(2) $F(\bar{p}) = \bar{F}(p)$. (The superbar denotes the complex conjugate of the unbarred symbol.)

(3) Real part $[F(i\omega)] \geq 0$ for all real values of ω.

These three conditions are necessary to insure that the function is the impedance of a possible linear, passive electric circuit structure. Interpreted physically in terms of this possible equivalent structure, the first condition specifies that the structure shall be stable; that is, every natural mode of oscillation dies away exponentially. The second condition specifies that the natural oscillations are real functions of time. The third condition specifies that if a sinusoidal current flows at the driving-point terminals of the equivalent structure, the average real power delivered to it will be positive. Since these three conditions, or their equivalents, are frequently mentioned in discussions of network theory, it is assumed that they are understood without more detailed explanation.

In addition to the above restrictions on the form of the impedance function, the following two conditions, while not necessary, will be imposed to limit the scope of the discussion:

(4) All the poles of $F(p)$ are simple.

(5) $F(p) = 0(1)$, exactly, as $|p| \rightarrow \infty$ everywhere except at the poles.

Condition (4), while limiting the scope of the exposition required, does not restrict the application of the results in any important way, because most impedance functions for which a network representation may be required have only simple poles.

Condition (5) implies that as p increases along any straight line drawn through the origin and not passing through any pole of $F(p)$, the modulus of $F(p)$ either approaches a limit or oscillates between finite limits. The physical implication of this condition is that the response of the network as a function of time to a suddenly applied cause begins with a discontinuity of the same degree as that of the cause. For example, the current response of the network to an applied step of voltage begins with a finite discontinuity. This behavior is a characteristic of continuous (non-lumped) electromagnetic structures, which furnish the principal application of the network developments to be described.

MITTAG-LEFFLER'S THEOREM[7]

Let the poles of the given function $F(p)$ be $p_1, p_2, p_3 \cdots$, where

$$0 < |p_1| \leq |p_2| \leq |p_3| \cdots$$

and let the residues at the poles be $A_1, A_2, A_3 \cdots$, respectively. Suppose that it is possible to draw a sequence of closed contours, C_n, such that C_n encloses $p_1, p_2, \cdots p_n$ but no other poles and such that the minimum distance of C_n from the origin tends to infinity with n. Suppose also that $F(p)$ satisfies conditions (2), (4) and (5) above. Then Mittag-Leffler's theorem gives the following series development for $F(p)$:

$$F(p) = F(0) + \underset{N\to\infty}{\text{Limit}} \sum_{n=-N}^{N} \left(\frac{A_n}{p - p_n} + \frac{A_n}{p_n}\right) \tag{1}$$

The notation here used employs the convention that

$$p_{-n} = \bar{p}_n \qquad \text{and} \qquad A_{-n} = \bar{A}_n,$$

since, by virtue of condition (2), the poles occur in conjugate complex pairs. The value, $n = 0$, then allows for a pole on the negative real axis.

Given any suitable function, the procedure is to determine its value for $p = 0$ and the location of its poles. The residues are next determined by

$$A_n = \underset{p\to p_n}{\text{Limit}} (p - p_n)F(p).$$

Then the Mittag-Leffler expansion can be written down at once.

NETWORK REPRESENTATION

In the series (1) the terms occur in pairs with conjugate complex poles and residues. The object is to obtain a network representation of each such pair of terms. If $F(p)$ is taken as an admittance, the branches representing the pairs of terms will all be connected in parallel; if $F(p)$ is taken as an impedance, they will all be connected in series.

Methods for obtaining a network representation for a rational function, such as the one comprising a pair of terms in the series (1), are well known. It is only necessary to describe certain procedures of particular application to the present problem. Brune[5] has stated that the necessary and sufficient condition for a network representation of a rational function of p to be realizable is that it be a "positive real function," that is, a function that is real for real values of p and whose real part is positive,

or zero, when the real part of p is positive, or zero. In view of conditions (1) and (2) above, only one test[12] need be applied to each pair of terms of the series (1): the sum of a pair of terms will be a positive real function if, and only if, the real part of their sum is greater than, or equal to, zero for all purely imaginary values of p.

The general term pair for which a network representation is sought is

$$F_n(p) = \frac{A_n}{p - p_n} + \frac{\bar{A}_n}{p - \bar{p}_n} + \frac{A_n}{p_n} + \frac{\bar{A}_n}{\bar{p}_n} = P_n(p) - P_n(0) \qquad (2)$$

Evidently two cases can be distinguished at the outset, depending upon whether $P_n(0)$ is positive or negative. If $P_n(0)$ is positive, the network branch, in order to be realizable, should be designed to represent $P_n(p)$. The left-over negative term, $-P_n(0)$, then can be absorbed in the positive first term, $F(0)$, of the series (1); more will be said of this later. If, on the other hand, $P_n(0)$ is negative, the network branch should represent the whole term, $P_n(p) - P_n(0)$. This procedure insures that the real part of the branch impedance will be positive, or zero, at zero and infinite frequencies. To guarantee that the resistance is positive at all other frequencies requires further tests now to be specified.

Let the real and imaginary coefficients of the poles and residues of the n^{th} term be

$$p_n = -\alpha_n + i\beta_n, \qquad \bar{p}_n = -\alpha_n - i\beta_n$$

$$A_n = a_n + ib_n, \qquad \bar{A}_n = a_n - ib_n$$

(With this notation, α_n and β_n are always positive; a_n and b_n can be either positive or negative.) Then (dropping the subscripts)

$$P(p) = \frac{2(a\alpha - b\beta) + 2ap}{\alpha^2 + \beta^2 + 2\alpha p + p^2}$$

$$R[P(i\omega)] = \frac{2(a\alpha - b\beta)(\alpha^2 + \beta^2) + 2\omega^2(a\alpha + b\beta)}{(\alpha^2 + \beta^2)^2 + 2\omega^2(\alpha^2 - \beta^2) + \omega^4} \qquad (3)$$

$$P(0) = \frac{2(a\alpha - b\beta)}{\alpha^2 + \beta^2}$$

$$R[P(i\omega) - P(0)] = \frac{-2(a\alpha^3 - 3\alpha^2 b\beta - 3a\alpha\beta^2 + b\beta^3)\omega^2 - 2(a\alpha - b\beta)\omega^4}{(\alpha^2 + \beta^2)[(\alpha^2 + \beta^2)^2 + 2(\alpha^2 - \beta^2)\omega^2 + \omega^4]}$$

The necessary and sufficient conditions[12] for the real part of a rational function of p to be positive, or zero, for purely imaginary values of p are that the function be positive for $p \to \pm i\infty$ and have no imaginary roots of odd multiplicity. When this test is applied to the functions $P(p)$ and

$P(p) - P(0)$, as given by (3), the following conditions are obtained: *$P(p)$ will be a positive real function if, and only if,*

$$a\alpha - b\beta > 0; \qquad \text{i.e.} \qquad P(0) > 0 \tag{4}$$

and

$$a\alpha + b\beta > 0$$

$P(p) - P(0)$ will be a positive real function if, and only if,

$$a\alpha - b\beta < 0; \qquad \text{i.e.} \qquad P(0) < 0 \tag{5}$$

and

$$a\alpha^3 - 3\alpha^2 b\beta - 3a\alpha\beta^2 + b\beta^3 < 0.$$

If all terms of the series satisfy one or the other of these conditions, network branches can be devised to represent all the terms and all the R, L, G, C elements of the branches will be positive.

In case all the terms are of the type where $P_n(0)$ is positive, so that the network branches are made to represent $P_n(p)$, the left-over constant terms can be collected and added to the first term, $F(0)$, of the series. This collection of terms then must be represented by a final branch of pure resistance, or conductance, of value,

$$F(0) - \sum_{n=o}^{\infty} P_n(0)$$

If the sum of the variable terms approaches zero for $p \rightarrow \pm i\infty$, the final constant term supplies the high frequency resistance of the function $F(p)$ and since this must be positive, if condition (3) is satisfied, the final resistive element will be positive. If the series converges non-uniformly, the sum of the variable terms can have a value other than zero as $p \rightarrow \pm i\infty$ in spite of the fact that every term approaches zero individually. In that case (see example 1) all or part of the high frequency resistance may be supplied by the sum of the variable terms.

In case all the terms are of the type where $P_n(0)$ is negative, so that the network branches are made to represent the sum, $P_n(p) - P_n(0)$, of the variable and constant terms and the series is uniformly convergent, all the high frequency resistance is provided by the branches representing these terms. The first term, $F(0)$ then supplies the dc resistance, which is positive by condition (3). Non-uniform convergence can modify this division of high- and low-frequency resistance, however.

Cases can arise in which the series contains terms of both types. In such a case the dc resistance, or high frequency resistance, or both, of

the given function might be less than the sum of the variable terms for these frequencies, with the result that the final resistance branch would be negative for either the series or parallel type of network development.

To make the procedure as concrete as possible, particular forms of networks are described in the section following with explicit formulas for computing their elements.

NETWORK FORMULAS

Simple forms of network branches are shown in Figs. 1 and 2. Those of Fig. 1, referred to as branches of "the first kind" are suitable for connection in parallel where the given function $F(p)$ is an admittance, $Y(p)$, while networks of "the second kind," shown in Fig. 2, are suitable for connection in series to represent an impedance, $F(p) = Z(p)$. The networks of Figs. 1a and 2a apply where the value $P_n(0)$ of the general term is positive, while Figs. 1b and 2b apply where $P_n(0)$ is negative. Figs. 3 and 4 illustrate, respectively, networks of the types of Figs. 1a and 2a

(a) (b)

Fig. 1—General branches of the first kind.

Fig. 1a	*Fig. 1b*	
(use where $F(p) = Y(p)$ and $Y_n(0) > 0$)	(use where $F(p) = Y(p)$ and $Y_n(0) < 0$)	
$L_n = \dfrac{1}{2a_n}$	$L_n = \dfrac{\beta_n^2(\alpha_n^2 + \beta_n^2)^2(a_n^2 + b_n^2)}{2M^3}$	
$\dfrac{1}{L_nC_n} = \beta_n^2\left(\dfrac{b_n^2}{a_n^2} + 1\right)$	$\dfrac{1}{L_nC_n} = \dfrac{M^2}{\beta_n^2(a_n^2 + b_n^2)}$	
$\dfrac{G_n}{C_n} = \dfrac{1}{a_n}(a_n\alpha_n - b_n\beta_n)$	$G_nL_n = -\dfrac{a_n\alpha_n - b_n\beta_n}{M}$	(6)
$\dfrac{R_n}{L_n} = \dfrac{1}{a_n}(a_n\alpha_n + b_n\beta_n)$	$R_nC_n = \dfrac{N}{M(\alpha_n^2 + \beta_n^2)}$	
$G_o = Y(0) - \sum_{n=o}^{\infty} Y_n(0)$	$G_o = Y(0)$	

connected to form the completed network with the final non-reactive branch, G_o or R_o, in place.

Formulas for the network elements are obtained by equating the poles and residues of the network impedance function to the given poles and residues of the general term of the series. Since both poles and residues occur in conjugate complex pairs, and since equality of real and imaginary parts is involved, there are four equations, which are necessary and sufficient to determine the four constants, R, L, G, C, of the network. The formulas that are obtained by solving these equations are given beneath Figs. 1 and 2.

The values given for G_o and R_o in each case assume that all the terms of the series are of the type specified for that case.

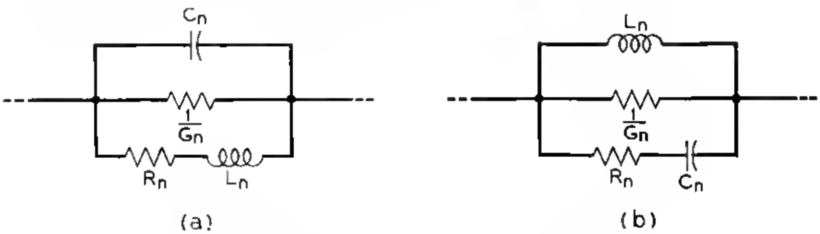


Fig. 2—General branches of the second kind.

Fig. 2a	*Fig. 2b*
(use where $F(p) = Z(p)$ and $Z_n(o) > 0$)	(use where $F(p) = Z(p)$ and $Z_n(o) < 0$)

$$
\begin{aligned}
C_n &= \frac{1}{2a_n} & \qquad C_n &= \frac{\beta_n^2(\alpha_n^2 + \beta_n^2)^2(a_n^2 + b_n^2)}{2M^3} \\
\frac{1}{L_nC_n} &= \beta_n^2\left(\frac{b_n^2}{a_n^2} + 1\right) & \qquad \frac{1}{L_nC_n} &= \frac{M^2}{\beta_n^2(a_n^2 + b_n^2)} \\
\frac{R_n}{L_n} &= \frac{1}{a_n}(a_n\alpha_n - b_n\beta_n) & \qquad R_nC_n &= -\frac{a_n\alpha_n - b_n\beta_n}{M} \\
\frac{G_n}{C_n} &= \frac{1}{a_n}(a_n\alpha_n + b_n\beta_n) & \qquad G_nL_n &= \frac{N}{M(\alpha_n^2 + \beta_n^2)} \\
R_o &= Z(0) - \sum_{n=0}^{\infty} Z_n(0) & \qquad R_o &= Z(0)
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\text{where } M &= a_n(\beta_n^2 - \alpha_n^2) + 2\alpha_n\beta_nb_n \\
N &= -a_n\alpha_n^3 + 3\alpha_n^2b_n\beta_n + 3a_n\alpha_n\beta_n^2 - b_n\beta_n^3
\end{aligned}
$$

In the case of the parallel-type networks (Figs. 1a and 1b), $p_n = -\alpha_n + i\beta_n$ is a pole of the admittance, $Y(p)$, and $A_n = a_n + ib_n$ is the corresponding residue. In the case of the series-type network, the same symbols represent a pole and residue of the impedance, $Z(p)$.

The networks specified by Figs. 2a and 2b are duals of the networks of Figs. 1a and 1b, respectively, and are obtained from the latter merely by replacing L_n by C_n, R_n by G_n, and vice versa.

The formulas are intended to apply to complex poles. They can be applied to real poles by taking b_n and β_n equal to zero and doubling the residue, a_n, but this procedure is unnecessary, because the network rep-

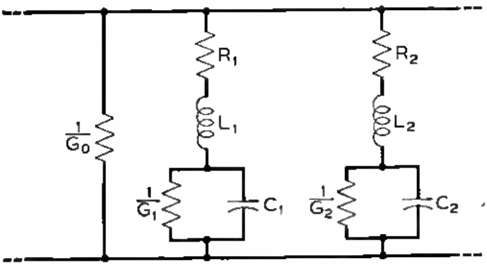


Fig. 3—Network of the first kind (branches 1a).

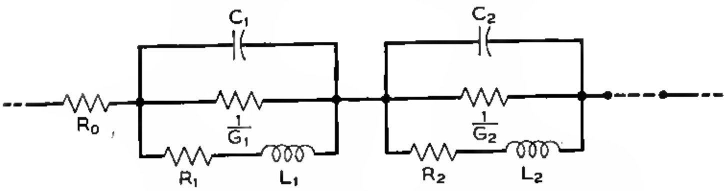


Fig. 4—Network of the second kind (branches 2a).

resentation of a real pole can be found readily enough by inspection of the impedance terms involved. (See Example 1.)

The above discussion is intended to sketch a general picture of the procedure. Individual cases may involve considerable detail that can be understood more readily by reference to the next section.

APPLICATIONS

Example 1a: A transmission line with its far terminals short-circuited affords a simple illustration of the equivalent network theory. Let it be assumed that the parameters, R, L, G and C of the line are constants. In the more advanced examples to follow, the variation of these parameters with frequency for a particular kind of line will be taken into consideration.

The impedance of the short-circuited line (Fig. 5) is

$$Z = Z_0 \tanh \Gamma \tag{1-0}$$

where Z_0 is the characteristic impedance and Γ is the total propagation constant of the line. We have

$$Z_0 = \left(\frac{R + pL}{G + pC}\right)^{1/2} \tag{1-1}$$

$$\Gamma = [(R + pL)(G + pC)]^{1/2} \tag{1-2}$$

R, L, G and C being given for the *total length* of line.

To obtain a development in terms of network branches of the kind shown in Fig. 1, we consider the admittance function,

$$Y = Y_0 \coth \Gamma \tag{1-3}$$

where $Y = 1/Z$ and $Y_0 = 1/Z_0$. Our first task is to find the poles of this function and the residues. Since the complex frequency variable p occurs

Fig. 5—Short-circuited transmission line.

under square roots in both Z_0 and Γ, it might be suspected, offhand, that the singularities of the function are branch points rather than poles. Such is not the case, however. There are no branch points and all the poles are simple.

The singularities of Y are to be found among the zeros of $\tanh \Gamma$, which occur at

$$\Gamma = i\pi n, \qquad n = 0, \pm 1, \pm 2, \pm 3, \cdots \tag{1-4}$$

To determine them, we solve

$$\Gamma^2 = (R + pL)(G + pC) = -\pi^2 n^2 \tag{1-5}$$

and find these roots:

$$p_n = -\alpha_n + i\beta_n, \qquad p_{-n} = \bar{p}_n = -\alpha_n - i\beta_n$$

where

$$\alpha_n = \frac{G}{2C} + \frac{R}{2L}$$

$$\beta_n = \left[\frac{\pi^2 n^2}{LC} - \left(\frac{G}{2C} - \frac{R}{2L}\right)^2\right]^{1/2} \qquad (n > 0) \tag{1-6}$$

For $n = 0$, the above would give

$$p_0 = -\frac{R}{L}, -\frac{G}{C}$$

But if we let $\Gamma \to 0$, so that $\tanh \Gamma \to \Gamma$, we find that only the point, $-R/L$, is a singularity of Y; the other point, $-G/C$, is a regular point. Therefore Y has only one real singularity.

To find the nature of the singularities of Y, we next calculate

$$\underset{p \to p_n}{\text{Limit}} \left[\frac{p - p_n}{Z_0(p) \tanh \Gamma(p)} \right] = A_n \tag{1-7}$$

and find that at each p_n the limit exists and has the value

$$A_n = \frac{1}{Z_0(p_n)\Gamma'(p_n)} = \frac{1}{L} - \frac{i\left(\alpha_n - \dfrac{R}{L}\right)}{\beta_n L} = a_n + ib_n \tag{1-8}$$

where $\Gamma'(p_n) = \dfrac{d}{dp}\Gamma(p)$, evaluated at $p = p_n$. The fact that this limit exists shows that all the singularities are simple poles. The values of A_n are then the residues at these poles.

When we now apply formulas (6) to determine the elements in the general branch of the equivalent network of Fig. 1a, we obtain, for $n > 1$,

$$L_n = \frac{L}{2}, \qquad \frac{1}{L_n C_n} = \frac{\pi^2 n^2}{LC}, \qquad \frac{G_n}{C_n} = \frac{G}{C}, \qquad \frac{R_n}{L_n} = \frac{R}{L}. \tag{1-9}$$

The network then comprises an infinite number of such branches in parallel. Each branch has the same elements R_n and L_n, equal, respectively, to half the total resistance and inductance of the transmission line, but the elements G_n and C_n decrease from one branch to the next in inverse proportion to the squares of the integers.

The Q of the n^{th} branch, which can be regarded as the Q of the associated resonance of the short-circuited line, is

$$Q_n = \frac{\omega_n}{2\alpha_n} = \frac{\omega_n}{\dfrac{G_n}{C_n} + \dfrac{R_n}{L_n}} = \frac{\omega_n}{\dfrac{G}{C} + \dfrac{R}{L}} \tag{1-10}$$

where

$$\omega_n = \sqrt{\frac{1}{L_n C_n} - \frac{G_n^2}{C_n^2}} = \sqrt{\frac{\pi^2 n^2}{LC} - \frac{G^2}{C^2}} \tag{1-11}$$

Thus, for small dissipation, the resonances would become sharper in direct proportion to the frequency (if the parameters R, L, G, C, were invariable with frequency, as assumed).

The above described branches of the equivalent network account only for the complex poles ($n > 1$) of the admittance function. Two more branches remain to be calculated. One is for the real pole ($n = 0$), which occurs at $p_0 = -R/L$, with residue, $A_0 = \frac{1}{L}$. The required branch for this pole is

$$\frac{A_0}{p - p_0} = \frac{1}{R + pL} \tag{1-12}$$

The other is the final conductance branch, which is calculated as follows:

$$G_0 = Y(0) + \sum_{n=-\infty}^{\infty} \frac{A_n}{p_n} = \sqrt{\frac{G}{R}} \coth \sqrt{GR} - \frac{1}{R} - 2G \sum_{n=-\infty}^{\infty} \frac{1}{\pi^2 n^2 + GR} = 0 \tag{1-13}$$

so that, for this example, the conductance branch vanishes. The network is drawn in Fig. 6.

A series type of network, as shown in Fig. 7, can be determined by

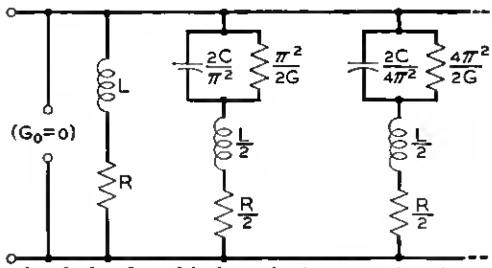


Fig. 6—Network of the first kind equivalent to the short-circuited line of Fig. 5.

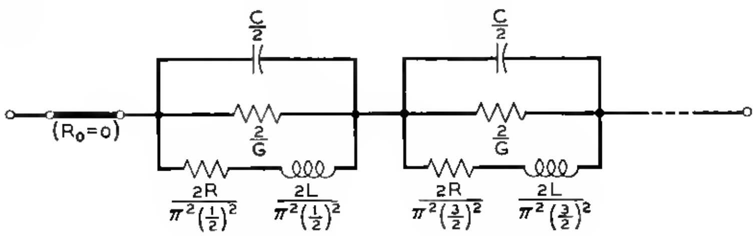


Fig. 7—Network of the second kind equivalent to the short-circuited line of Fig. 5.

similar means. Since, however, it is a dual of the parallel network of Fig. 9 for the open-circuited line, next to be discussed, it can be drawn immediately, without further calculation, once the latter has been found.

Example 1b: We now calculate a network for the same line with its far terminals open (Fig. 8). To obtain a network of the first kind, with branches in parallel, we deal with the admittance function,

$$Y = Y_0 \tanh \Gamma \tag{1-14}$$

The singularities of Y are found among the zeros of $\coth \Gamma$, which occur at

$$\Gamma = i\pi(n + \tfrac{1}{2}), \qquad n = 0, \pm 1, \pm 2, \pm 3, \cdots \tag{1-15}$$

The points $p = -R/L$ and $-G/C$ are *both* regular points this time. ($-G/C$ is a zero of Y.) The singularities are simple poles, as before, with residues,

$$A_n = \frac{1}{Z_0(p_n)\Gamma'(p_n)} \tag{1-16}$$

as before.

The network branches for the complex poles are therefore obtained merely by putting $n + \frac{1}{2}$ in place of the n in all formulas of the short-circuit network. There is no branch corresponding to the branch $R + pL$ of the other network and the conductance branch is again found to be zero. The complete parallel network is drawn in Fig. 9 and the series network, in Fig. 10.

It will be observed that the series network of Fig. 10 is the dual of the

Fig. 8—Open-circuited transmission line.

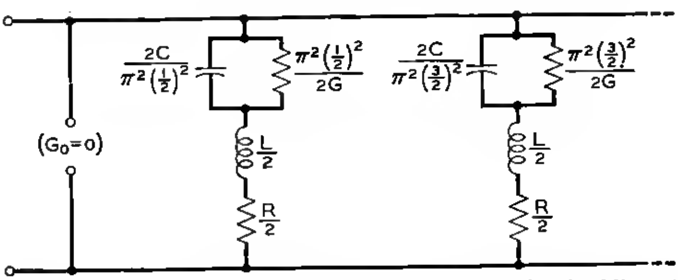


Fig. 9—Network of the first kind equivalent to the open-circuited line of Fig. 8.

parallel network of Fig. 6 and the series network of Fig. 7 is the dual of the parallel network of Fig. 9. These dual relationships are of course a result of the fact that the impedance of an open-circuited line is the dual of the impedance of the same line when short-circuited.

Example 2: Short-circuited Concentric Line (or Toroidal Cavity with E Radial). The preceding example considered a fictitious transmission line of invariable parameters, R, L, G, C, having a perfect short circuit at one end. The present example has to do essentially with the same problem but considers it from a more practical point of view. The variation of R and L with frequency is taken into account and the impedance of the "short-circuit" is no longer neglected.

Let the line be the piece of coaxial cable plugged at both ends with conducting material as illustrated in Fig. 11. Considered from an alternative point of view, our line is now a toroidal cavity oscillating in the

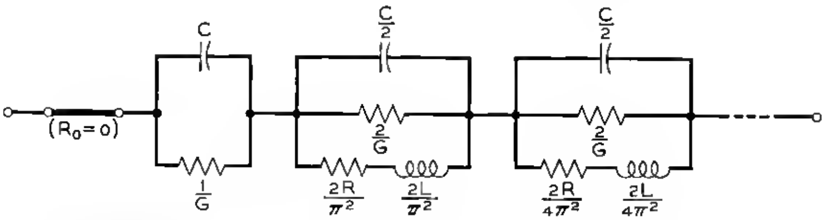


Fig. 10—Network of the second kind equivalent to the open-circuited line of Fig. 8.

mode where the electric force E is directed radially and the magnetic force H lies in planes at right angles to the axis. If we assume the cavity to be excited, or "driven," from one end,* the impedance that is effective in defining the selective characteristic of the cavity with respect to frequency is the total impedance at that end, that is, the sum of the impedance Z_1, viewed into the cavity, and the impedance, Z_2, of the adjacent end-plug. Therefore, we have to deal with the impedance,

$$Z = Z_1 + Z_2 . \tag{2-1}$$

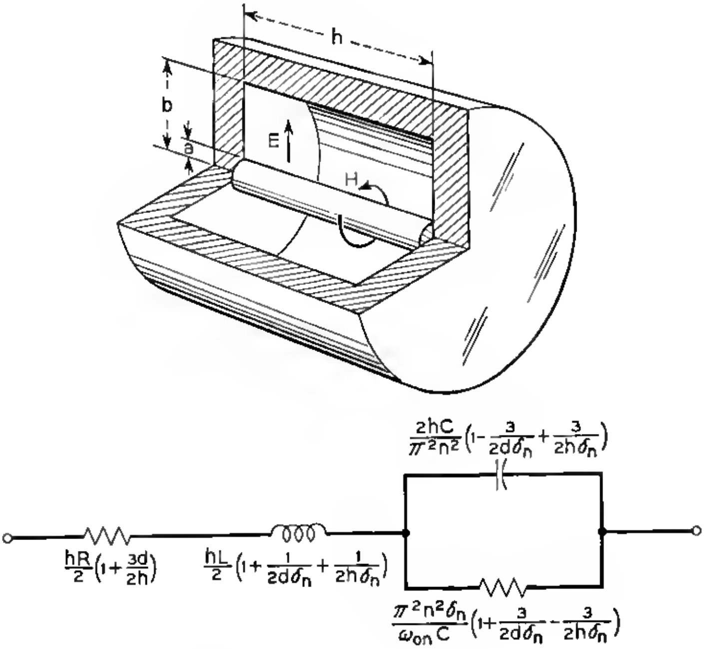
By "impedance" is here meant the same thing that one considers in looking at the problem from the point of view of transmission line theory, namely, the complex ratio, for exponential oscillations, of the voltage between the inside and outside cylindrical surfaces to the total current

* For determining the "natural frequencies" of oscillation of the cavity, it is immaterial at what point along it the impedance is taken; the total impedance at every point has the same roots. The impedance is, nevertheless, not the same at all points so that the behavior of the cavity, when driven, will depend to some extent on the driving point.

flowing axially in the inner conductor at the same point. The zeros of Z define the natural frequencies of oscillation of the cavity and their associated damping constants, or Q's. Our task is to develop an equivalent network for this Z.

We have

$$Z = Z_1 + Z_2 = Z_0 \left(\frac{1 + \rho e^{-2\gamma h}}{1 - \rho e^{-2\gamma h}} + \frac{1 + \rho}{1 - \rho} \right) \tag{2-2}$$



$$R = \frac{1}{2\pi}\sqrt{\frac{\omega_{on}\mu}{2g}}\left(\frac{1}{a} + \frac{1}{b}\right) \qquad \omega_{on} = \frac{\pi n v}{h}$$

$$L = \frac{\mu_0}{2\pi} \text{LOG} \frac{b}{a} \qquad v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3\,(10^8)$$

$$C = \frac{2\pi\epsilon_0}{\text{LOG} \frac{b}{a}} \qquad \delta_n = \sqrt{\frac{\omega_{on}\mu g}{2}}$$

(e.g.) $\mu_0 = 4\pi\,(10^{-7})$, $\epsilon_0 = \frac{10^{-9}}{36\pi}$, FOR AIR IN M.K.S. UNITS

$\mu = \mu_0$, $g = 5.8\,(10^7)$, FOR COPPER IN M.K.S. UNITS

($n = 1$ FOR FUNDAMENTAL MODE)

Fig. 11—Toroidal cavity, E radial.

where

$$Z_0 = \left(\frac{R + pL}{G + pC}\right)^{1/2}, \qquad \gamma = (R + pL)^{1/2}(G + pC)^{1/2} \tag{2-3}$$

$$\rho = \frac{Z_2 - Z_0}{Z_2 + Z_0} \tag{2-4}$$

$$Z_2 = \frac{\eta}{2\pi} \log \frac{b}{a} \tag{2-5}$$

$$R + pL = \frac{\eta}{2\pi a} \frac{I_0(\sigma a)}{I_1(\sigma a)} + \frac{\eta}{2\pi b} \frac{K_0(\sigma b)}{K_1(\sigma b)} + \frac{p\mu_0}{2\pi} \log \frac{b}{a} \tag{2-6}$$

$$G + pC = \frac{2\pi}{\log \frac{b}{a}} (g_0 + p\epsilon_0) \tag{2-7}$$

h, a, b = cavity length, inner radius, outer radius, as shown in Fig. 11, all measured in meters

$$\eta = \left(\frac{p\mu}{g}\right)^{1/2} \qquad \sigma = (p\mu g)^{1/2} \tag{2-8}$$

μ, g are permeability, conductivity of the conducting material of the walls (for copper: $\mu = 4\pi(10^{-7})$, $g = 5.8(10^7)$ in M.K.S. units).

μ_0, g_0, ϵ_0 are permeability, conductivity, dielectric constant of the dielectric material occupying the cavity (for air: $\mu_0 = 4\pi(10^{-7})$, $g_0 = 0$, $\epsilon_0 = (10^{-9})/36\pi$ in M.K.S. units), p = generalized frequency variable.

$I_0(z)$, $I_1(z)$ are Bessel functions of the first kind for imaginary argument and of order 0, 1.

$K_0(z)$, $K_1(z)$ are Bessel functions of the second kind for imaginary argument and of order 0, 1.

Except for ignored small deviations of the field around the corners of the cavity, the above formulas are exact. To arrive at results that are sufficiently compact to be useful, we make these approximations, at the start:

$$Z_0 = K_0 = \left[\frac{L}{C}\right]^{1/2} = \frac{\eta_0}{2\pi} \log \frac{b}{a}, \tag{2-9}$$

where

$$\eta_0 = \left[\frac{\mu_0}{\epsilon_0}\right]^{1/2} = 120\pi \text{ ohms} \tag{2-10}$$

From this,

$$\rho = \frac{\eta - \eta_0}{\eta + \eta_0} \tag{2-11}$$

Having in mind microwave applications, where the moduli of the arguments of the Bessel functions are >3000, we take

$$\frac{I_0(z)}{I_1(z)} = \frac{K_0(z)}{K_1(z)} = 1$$

so that

$$R + pL = \frac{\eta}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right) + \frac{p\mu_0}{2\pi}\log\frac{b}{a} \tag{2-12}$$

Also, we have in mind only air dielectric and assume any loss therein to be negligible; that is, we assume $G = 0$.

All further approximations that are made are either

$$\frac{1}{1 - \Delta} \doteq 1 + \Delta \qquad \text{or} \qquad (1 + 2\Delta)^{1/2} \doteq 1 + \Delta$$

where, for an air-space enclosed by copper walls, and for frequencies on the order of 30,000 megacycles, Δ is on the order of 10^{-4}. For cavities made of other materials, the results obtained may not be sufficiently accurate and the problem would have to be reviewed from the start. In particular, the results do not hold for a cavity having walls of magnetic material, because we assume here that the permeability of the metal walls is the same as that of air; i.e., $\mu = \mu_0$.

To obtain an equivalent network of the first kind, we deal with the admittance, which is, from (2-2),

$$Y = \frac{1}{Z} = H_0\,\frac{(1 - \rho)(1 - \rho e^{-2\gamma h})}{2(1 - \rho^2 e^{-2\gamma h})} \tag{2-13}$$

where $H_0 = 1/K_0$.

The poles of Y are then the zeros of $1 - \rho^2 e^{-2\gamma h}$, which are obtained by successive approximations. We first make a close estimate of the zeros by assuming that the impedance of the short-circuiting plugs is zero; that is, we assume, $Z_2 = 0$, whence $\rho = -1$. To obtain this estimate, we have to solve

$$\gamma h = \frac{ph}{v}\left(1 + \frac{2}{d\sigma}\right)^{1/2} = \pi i n \qquad (n = \pm 1, \pm 2, \pm 3 \cdots) \tag{2-14}$$

where

$$d = \frac{2ab \log (b/a)}{a + b}$$

and $v = 3(10^8)$ meters per second. The approximate solution is

$$p_{1n} = p_{0n}\left(1 + \frac{1}{d\sigma_{0n}}\right)$$

where

$$p_{0n} = \frac{i\pi nv}{h} \qquad \text{and} \qquad \sigma_{0n} = (p_{0n}\mu g)^{1/2}$$

Next we improve our estimate of the zeros by the well-known method involving the derivative of the function, $1 - \rho^2 e^{-2\gamma h}$, with respect to p, evaluated at p_{1n}. This now takes account of the actual impedance of the end-plugs. The values of the zeros, so obtained, are

$$p_n = -\alpha_n + i\beta_n, \qquad p_{-n} = \bar{p}_n = -\alpha_n - i\beta_n$$

where

$$\begin{aligned} \alpha_n &= \omega_{0n}\left(\frac{1}{2d\delta_n} + \frac{1}{h\delta_n}\right) \\ \beta_n &= \omega_{0n}\left(1 + \frac{1}{2d\delta_n} - \frac{1}{h\delta_n}\right) \end{aligned} \tag{2-15}$$

where δ_n* is the real part of σ_{0n}. That is,

$$\delta_n = (\omega_{0n}\mu g/2)^{1/2}$$

where

$$\omega_{0n} = \frac{\pi nv}{h}.$$

As an incidental matter of interest, the above gives the Q of the cavity at any resonance, namely

$$Q_n = \frac{\beta_n}{2\alpha_n} = d\delta_n \frac{1}{1 + \dfrac{2d}{h}} \tag{2-16}$$

For example, the dimensions, $a = .5$ cm., $b = 1.0$ cm., $h = .5$ cm. provide a cavity that resonates at about 30,000 megacycles. Then the Q's at the first three resonances would be as follows:

n	$\dfrac{\omega_{0n}}{2\pi}$	Q
1	$30{,}000 \times 10^6$	4250
2	$60{,}000 \times 10^6$	6010
3	$90{,}000 \times 10^6$	7360

* For any frequency, $\delta = (\omega\mu g/2)^{1/2}$ is sometimes referred to as the "skin depth" because it is the depth of metal at which the current density falls to $1/e$ times its value at the surface of the metal.

The importance of including the effect of the end-plugs in determining Q is shown by the fact that, if they were assumed to have zero impedance, Q at the first resonance would be 12,120 instead of 4250.

To determine the residues at the poles, we write

$$Y = H_0 \frac{(1 - \rho)(1 - \rho e^{-2\gamma h})}{2(1 - \rho^2 e^{-2\gamma h})} = \frac{F(p)}{G(p)} \tag{2-17}$$

and then the residue at a simple pole p_n is

$$A_n = \frac{F(p_n)}{G'(p_n)} \tag{2-18}$$

This limit is found to exist, showing that the poles are, in fact, simple. The value found for the residue, A_n, is

$$\begin{aligned} A_n &= a_n + ib_n, \qquad A_{-n} = \bar{A}_n = a_n - ib_n \\ a_n &= \frac{H_0\omega_{0n}}{\pi n}\left(1 - \frac{1}{2d\delta_n} - \frac{1}{2h\delta_n}\right) \\ b_n &= \frac{H_0\omega_{0n}}{\pi n}\left(\frac{1}{2d\delta_n} + \frac{1}{2h\delta_n}\right) \end{aligned} \tag{2-19}$$

When formulas (6) are applied to determine the elements of the tuned branches of the equivalent network of the first kind, the results are, for the n^{th} branch,

$$\begin{aligned} L_n &= \frac{K_0\pi n}{2\omega_{0n}}\left(1 + \frac{1}{2d\delta_n} + \frac{1}{2h\delta_n}\right) \\ \frac{1}{L_nC_n} &= \omega_{0n}^2\left(1 + \frac{1}{d\delta_n} - \frac{2}{h\delta_n}\right) \\ \frac{G_n}{C_n} &= \frac{\omega_{0n}}{2h\delta_n} \\ \frac{R_n}{L_n} &= \omega_{0n}\left(\frac{1}{d\delta_n} + \frac{3}{2h\delta_n}\right) \end{aligned} \tag{2-20}$$

In terms of the R, L and C of the piece of coaxial line, the elements of the n^{th} branch are as follows:

$$\begin{aligned} L_n &= \frac{hL}{2}\left(1 + \frac{1}{2d\delta_n} + \frac{1}{2h\delta_n}\right) \\ R_n &= \frac{hR}{2}\left(1 + \frac{3d}{2h}\right) \\ C_n &= \frac{2hC}{\pi^2 n^2}\left(1 - \frac{3}{2d\delta_n} + \frac{3}{2h\delta_n}\right) \\ G_n &= \frac{\omega_{0n}C}{\pi^2 n^2 \delta_n}\left(1 - \frac{3}{2d\delta_n} + \frac{3}{2h\delta_n}\right) \end{aligned} \tag{2-21}$$

where

$$L = \frac{\mu_0}{2\pi} \log \frac{b}{a}, \qquad R = \frac{1}{2\pi}\left(\frac{\omega_{0n}\mu}{2g}\right)^{1/2}\left(\frac{1}{a} + \frac{1}{b}\right)$$

$$C = \frac{2\pi\epsilon_0}{\log \dfrac{b}{a}}, \qquad \omega_{0n} = \frac{\pi n v}{h}, \qquad \delta_n = \left(\frac{\omega_{0n} n \mu g}{2}\right)^{1/2}$$

The network is shown in Fig. 11.

It will be found that a "leakage" element, G_n, appears in the equivalent network, although the air dielectric in the cavity was assumed to have no leakage ($G = 0$). This element arises from the end-plugs and is necessary to account for the dissipation in them.

To obtain a network exactly equivalent to the cavity at all frequencies, we should add a branch corresponding to $n = 0$, as was done in example 1. This branch would make the equivalence hold down to and including zero frequency. But, inasmuch as the approximations that have been made hold only for the high frequencies, where the resonances occur, it would be inconsistent to add this branch. What has been arrived at, then, is a partial network representation that gives a close approximation to the impedance of the cavity at high frequencies, only.

Example 3: Toroidal Cavity with E Axial. For further illustration, we consider another mode of oscillation of the short-circuited concentric transmission line investigated in the previous example. This time it is assumed that the radial electric force vanishes while the axial electric force between the end-plugs exists. The magnetic force is directed in circles concentric with the cylindrical central conductor, as before. This situation is illustrated in Fig. 12, which is the same as Fig. 11, except for the new disposition of the E-vector.

For the new mode of oscillation, where the wave is a cylindrical one propagated back and forth between the inner and outer conducting cylinders, the oscillatory space is naturally thought of as a "toroidal cavity," while, in the previous example, where the wave was propagated axially back and forth between the terminal discs, the space was called a "concentric line." Actually, the cavity itself has the same geometric form in the two cases. A practical distinction may exist, however, in that the axial mode of oscillation could be more easily excited in a cavity whose axial length is large compared to its radius, while the cylindrical mode would arise more easily in a flat "pillbox" cavity whose radius is large compared to its axial dimension.

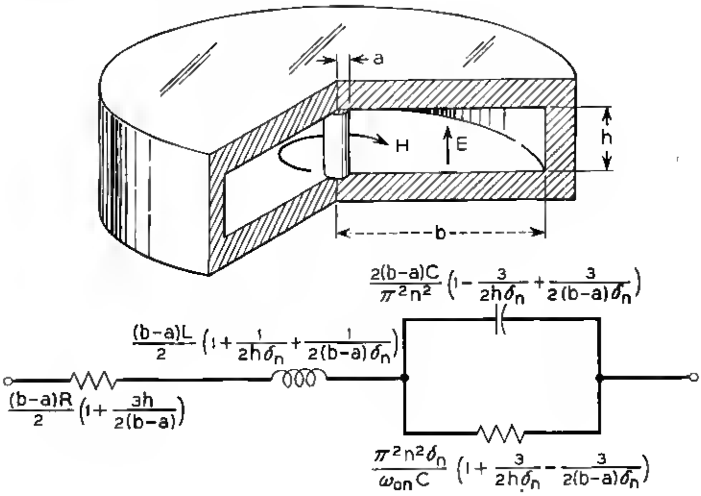
The approach to the problem will be that of transmission line theory, as before. This time, the "line" comprises two circular discs between

which the cylindrical wave is propagated. The series impedance and shunt admittance of such a line are functions of the radius and so will be designated $Z(r)$ and $Y(r)$, respectively. Their values are given below:

$$Z(r) = \frac{2\eta + i\omega\mu h}{2\pi r} \tag{3-1}$$

$$Y(r) = \frac{i\omega 2\pi r\epsilon_0}{h} \tag{3-2}$$

These formulas take into account the losses in the flat walls but assume the conductance of the air between them to be negligible. Losses in the inner and outer "short-circuiting" cylinders will be taken into account by the boundary conditions.



$R = \frac{1}{\pi a}\sqrt{\frac{\omega_{on}\mu}{2g}}$ $\qquad$ $\omega_{on} = \frac{\pi n v}{b-a}$

$L = \frac{\mu_0 h}{2\pi a}$ $\qquad$ $v = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 3(10^8)$

$C = \frac{2\pi a\epsilon_0}{h}$ $\qquad$ $\delta_n = \sqrt{\frac{\omega_{on}\mu g}{2}}$

SEE FIGURE 11 FOR $\mu_0, \epsilon_0, \mu, g$

($n=1$ FOR FUNDAMENTAL MODE)

Fig. 12—Toroidal cavity, E axial.

If V is the voltage between the flat faces of the cavity at a radius r and I the total current in the lower face at this radius, we have

$$\begin{aligned} \frac{dV}{dr} &= -IZ(r) \\ \frac{dI}{dr} &= -VY(r) \end{aligned} \tag{3-3}$$

By differentiating,

$$\frac{d^2V}{dr^2} = -I\frac{dZ}{dr} - Z\frac{dI}{dr} = \left(\frac{1}{Z}\frac{dZ}{dr}\right)\frac{dV}{dr} + VZY \tag{3-4}$$

But

$$ZY = (2\eta + i\omega\mu h)\frac{i\omega\epsilon_0}{h} = \gamma^2$$

which is a squared propagation constant, *independent of* r, and

$$\frac{1}{Z}\frac{dZ}{dr} = -\frac{1}{r}$$

Therefore,

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \gamma^2 V = 0 \tag{3-5}$$

is the differential equation for the voltage. The usual solution of this equation is a linear combination of $I_0(\gamma r)$ and $K_0(\gamma r)$ but since, in this case, the arguments will be almost purely imaginary, it is more convenient to employ the pair of functions, $J_0(-i\gamma r)$ and $N_0(-i\gamma r)$.

The solution for the voltage between the upper and lower surfaces at radius r is

$$V(r) = AJ_0(-i\gamma r) + BN_0(-i\gamma r) \tag{3-6}$$

and, from this, the total radial current in the lower surface, at that radius, is

$$I(r) = -\frac{1}{Z}\frac{dV(r)}{dr} = -iY_0(r)[AJ_1(-i\gamma r) + BN_1(-i\gamma r)] \tag{3-7}$$

where

$$Y_0(r) = 1/Z_0(r) = [Y(r)/Z(r)]^{1/2}$$

The impedance at the inner radius a, looking outward, is then

$$Z_1(a) = \frac{V(a)}{I(a)} = iZ_0(a)\frac{AJ_0(-i\gamma a) + BN_0(-i\gamma a)}{AJ_1(-i\gamma a) + BN_1(-i\gamma a)} \tag{3-8}$$

The total impedance at a (inward + outward) for which we require an equivalent network is

$$Z = Z_1(a) + Z_a$$

where Z_a is the impedance of the central plug to axial current, viz.,

$$Z_a = \frac{\eta h}{2\pi a} \frac{I_0(\sigma a)}{I_1(\sigma a)} \tag{3-9}$$

To evaluate the constants A and B, the following boundary conditions are imposed at radii a and b:

$$\text{at } a\text{: } V = V(a), \text{ a given voltage}$$

$$\text{at } b\text{: } V = I(b)Z_b$$

where Z_b is the impedance of the other "short circuit," comprising the outer cylindrical wall. It is given by

$$Z_b = \frac{\eta h}{2\pi b} \frac{K_0(\sigma b)}{K_1(\sigma b)} \tag{3-10}$$

Except for ignored small deviations of the field around the corners of the cavity, the above expressions are exact. The process of finding the singularities of Z by successive approximations results in expressions that are too long to write down here. To obtain results sufficiently compact for engineering use, we resort to the following asymptotic approximations for the Bessel functions:

$$\begin{aligned}
J_0(z) &\sim \left(\frac{2}{\pi z}\right)^{1/2} \cos(z - \pi/4) \\
J_1(z) &\sim \left(\frac{2}{\pi z}\right)^{1/2} \cos(z - 3\pi/4) \\
N_0(z) &\sim \left(\frac{2}{\pi z}\right)^{1/2} \sin(z - \pi/4) \\
N_1(z) &\sim \left(\frac{2}{\pi z}\right)^{1/2} \sin(z - 3\pi/4) \\
\frac{I_0(z)}{I_1(z)} &\sim 1, \qquad \frac{K_0(z)}{K_1(z)} \sim 1
\end{aligned} \tag{3-11}$$

Also, with an error on the order of 10^{-4},

$$Z_0(r) \sim \frac{h\eta_0}{2\pi r} = K_0(r) = 1/H_0(r)$$

These substitutions result in the following asymptotic formula for the total impedance Z at radius a

$$Z = K_0(a) \frac{\dfrac{2\eta}{\eta_0} \cos kx + i\left(1 + \dfrac{\eta^2}{\eta_0^2}\right) \sin kx}{\cos kx + \dfrac{i\eta}{\eta_0} \sin kx} \tag{3-12}$$

where $k = \dfrac{b}{a} - 1$ and $x = -i\gamma a$.

To find an equivalent network of the first kind to represent Z, we deal with the admittance, $Y = 1/Z$. It is instructive and saves much work to put Y in the form of exponential functions, with the substitution

$$\rho = \frac{\eta - \eta_0}{\eta + \eta_0}$$

which is the reflection coefficient at both inside and outside cylindrical surfaces of the cavity. By this means we obtain

$$Y = H_0(a) \frac{(1 - \rho)(1 - \rho e^{-2ikx})}{2(1 - \rho^2 e^{-2ikx})} \tag{3-13}$$

This is now identical in form to the formula (2-13) of example 2, where the E-vector was radially, instead of axially, directed. In fact, since

$$ikx = \gamma(b - a)$$

and

$$\gamma = \frac{i\omega}{v}\left(1 + \frac{2}{h\sigma}\right)^{1/2}$$

comparison with the similar formulas of example 2 shows that all the results of that example can be made to apply to the present one merely by changing the dimensional parameters as follows:

Example 2 (*E radial*)		Example 3 (*E axial*)
h	goes into	$b - a$
$d = \dfrac{2ab \log (b/a)}{a + b}$	goes into	h

The first result of interest is the value of Q, which is

$$Q_n = h\delta_n \frac{1}{1 + \dfrac{2h}{b - a}} \tag{3-14}$$

where, as before, δ_n is the "skin depth" equal to the real part of σ_n. That is,

$$\delta_n = \sqrt{\frac{\omega_{0n}\mu g}{2}}$$

To gain an idea of numerical magnitudes, consider the same cavity used in example 2. The dimensions are, as before, $h = .5$ cm., $b - a = .5$ cm. For the square cross-section chosen, the first resonance again occurs at 30,000 megacycles, very nearly, and we can make the following direct comparison of the Q's for the two modes of oscillation:

n	$\omega_{0n}/2\pi$	Q_n	
		Ex. 2 (E radial)	Ex. 3 (E axial)
1	$30,000 \times 10^6$	4250	4370
2	$60,000 \times 10^6$	6010	6180
3	$90,000 \times 10^6$	7360	7560

Due to the asymptotic approximations used, the results for example 3 are not as accurate as those for example 2; the two sets of results show only that the Q of the cavity is substantially the same for the two different modes of oscillation.

The poles of Y are given by

$$p_n = -\alpha_n + i\beta_n, \qquad p_{-n} = \bar{p}_n = -\alpha_n - i\beta_n$$

$$\alpha_n = \omega_{0n}\left[\frac{1}{2h\delta_n} + \frac{1}{(b-a)\delta_n}\right] \tag{3-15}$$

$$\beta_n = \omega_{0n}\left[1 + \frac{1}{2h\delta_n} - \frac{1}{(b-a)\delta_n}\right]$$

and the residues are

$$A_n = a_n + ib_n, \qquad A_{-n} = \bar{A}_n = a_n - ib_n$$

$$a_n = \frac{H_0(a)\omega_{0n}}{\pi n}\left[1 - \frac{1}{2h\delta_n} - \frac{1}{2(b-a)\delta_n}\right] \tag{3-16}$$

$$b_n = \frac{H_0(a)\omega_{0n}}{\pi n}\left[\frac{1}{2h\delta_n} + \frac{1}{2(b-a)\delta_n}\right]$$

Applying formulas (6) gives the following values for the n^{th} branch of

the network of the first kind:

$$
\begin{aligned}
L_n &= K_0(a)\,\frac{\pi n}{2\omega_{0n}}\left[1 + \frac{1}{2h\delta_n} + \frac{1}{2(b-a)\delta_n}\right] \\
\frac{1}{L_nC_n} &= \omega_{0n}^2\left[1 + \frac{1}{h\delta_n} - \frac{2}{(b-a)\delta_n}\right] \\
\frac{G_n}{C_n} &= \frac{\omega_{0n}}{2(b-a)\delta_n} \\
\frac{R_n}{L_n} &= \omega_{0n}\left[\frac{1}{h\delta_n} + \frac{3}{2(b-a)\delta_n}\right]
\end{aligned}
\tag{3-17}
$$

in all of which $\omega_{0n} = \pi n v/(b - a)$ and $v = 1/(\mu_0\epsilon_0)^{1/2} = 3(10^8)$ meters per second.

The results can be put in the same form as those obtained for the other cavity mode, dealt with in example 2, by employing the "primary constants" of the cylindrical transmission line, viz.:

$$
R(a) = \frac{1}{\pi a}\left[\frac{\omega_{0n}\mu}{2g}\right]^{1/2} \qquad L(a) = \frac{\mu h}{2\pi a}
$$

$$
G(a) = 0 \qquad C(a) = \frac{2\pi a\epsilon_0}{h}
$$

In terms of these constants, the elements of the n^{th} branch of the equivalent network of the first kind are

$$
\begin{aligned}
L_n &= \frac{(b-a)L(a)}{2}\left(1 + \frac{1}{2h\delta_n} + \frac{1}{2(b-a)\delta_n}\right) \\
R_n &= \frac{(b-a)R(a)}{2}\left(1 + \frac{3h}{2(b-a)}\right) \\
C_n &= \frac{2(b-a)C(a)}{\pi^2n^2}\left(1 - \frac{3}{2h\delta_n} + \frac{3}{2(b-a)\delta_n}\right) \\
G_n &= \frac{\omega_{0n}C(a)}{\pi^2n^2\delta_n}\left(1 - \frac{3}{2h\delta_n} + \frac{3}{2(b-a)\delta_n}\right)
\end{aligned}
\tag{13-8}
$$

The network is shown in Fig. 12.

As in the preceding example, a leakage element arises, in spite of the fact that we assumed initially that g_0 of the air in the cavity is zero. This element accounts for the losses in the inner and outer cylindrical walls.

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